arXiv:math/0609221v1 [math.CO] 8 Sep 2006

On Ear Decompositions of Strongly Connected Bidirected Graphs

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February 2, 2008

Abstract

Bidirected graphs (earlier studied by Edmonds, Johnson and, in equivalent terms of skew-symmetric graphs, by Tutte, Goldberg, Karzanov, and others) proved to be a useful unifying language for describing both flow and matching problems. In this paper we extend the notion of ear decomposition to the class of strongly connected bidirected graphs. In particular, our results imply Two Ear Theorem on matching covered graphs of Lovász and Plummer. The proofs given here are self-contained except for standard Barrier Theorem on skew-symmetric graphs.

Keywords: bidirected graph, skew-symmetric graph, strong connectivity, ear decomposition.

AMS Subject Classification: 05C38, 05C40, 05C75.

1 Introduction

For an arbitrary undirected graph G we write V_G (resp. E_G) to denote the set of nodes (resp. edges) of G. In case G is directed we speak of arcs rather than edges and write A_G instead of E_G. The same notation will be used for walks, paths, cycles etc.

Consider a digraph G and its arbitrary subgraph H (that is, $V_H \subseteq V_G$, $A_H \subseteq A_G$). An *ear* of H w.r.t. G is a path P in G such that: (i) both ends of P are in V_H; (ii) no inner node of P is in V_H; (iii) $A_P \cap A_H = \emptyset$. In particular, an ear can consist of a single arc a with both head and tail nodes in V_H; as long as this is not confusing we denote this ear by a. By $H' := H + P$ we denote a new digraph with $V_{H'} := V_H \cup V_P$, $A_{H'} := A_H \cup A_P$. Also, for a collection $\mathcal{P}$ of ears we denote by $G + \mathcal{P}$ the result of adding all ears from $\mathcal{P}$ to G.

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Recall that a digraph G is called *strongly connected* if for any pair of nodes in G the former one is reachable from the latter by a path or, equivalently, the underlying undirected graph of G is connected and each arc of G is contained in a cycle.

For a pair of strongly connected digraphs G, H, where H is a subgraph of G, we define an *ear decomposition* of G starting from H to be a sequence of strongly connected subgraphs of G

$$H = G_0, G_1, \ldots, G_{k-1}, G_k = G,$$

where G_{i+1} is obtained from G_i by adding an ear of G_i w.r.t. G $(0 \le i < k)$. Clearly, an ear decomposition is not unique.

Remark 1.1 *One can easily see that the requirement for $G_1, \ldots, G_k$ to be strongly connected can be dropped since adding an ear to a strongly connected digraph preserves strong connectivity. This will not be the case for the class of bidirected graphs so we keep this requirement to make our definitions more symmetric.*

A central fact about ear decompositions of strongly connected digraphs is stated in the next folklore theorem:

Theorem 1.2 *For any strongly connected digraph G and an arbitrary strongly connected subgraph H of G there exists an ear decomposition of G starting from H.*

The main goal of this paper is to extend the notion of ear decomposition and Theorem 1.2 to the class of *bidirected* graphs. It turns out that this generalization will naturally contain certain well-known decomposition results from matching theory.

The notion of bidirected graphs was introduced by Edmonds and Johnson [3] in connection with one important class of integer linear programs generalizing problems on flows and matchings; for a survey, see also [6, 8].

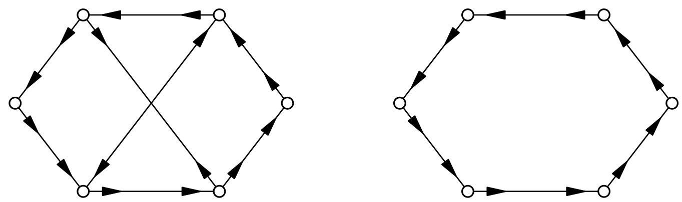
Recall that in a *bidirected* graph G three types of edges are allowed: (i) a standard directed edge, or an *arc*, that leaves one node and enters another one; (ii) a nonstandard edge leaving both of its ends; or (iii) a nonstandard edge entering both of its ends.

When both ends of an edge coincide, the edge becomes a *loop*.

We borrow the notation that was introduced for undirected graphs and write V_G (resp. E_G) to denote the set of nodes (resp. edges) of a bidirected graph G.

A *walk* in a bidirected graph G is an alternating sequence $P = (s = v_0, e_1, v_1, \ldots, e_k, v_k = t)$ of nodes and edges such that each edge e_i connects nodes v_{i-1} and v_i, and for $i = 1, \ldots, k-1$, the edges e_i, e_{i+1} form a *transit pair* at v_i, which means that one of e_i, e_{i+1} enters and the other leaves v_i. Note that e_1 may enter s and e_k may leave t; nevertheless, we refer to P as a walk from s to t, or an *s–t walk*. P is *cyclic* if $v_0 = v_k$ and the pair e_1, e_k is transit at v_0; cyclic walks are usually considered up to cyclic shifts. Observe that an s–s walk is not necessarily cyclic.

A walk is called *edge-simple* (or a *path*) if all its edges are different. If $v_i \ne v_j$ for all $1 \le i < j < k$ and $1 < i < j \le k$, then walk P is called *node-simple* (or a *simple* path). Note



(a) Graph G. (b) Subgraph H.

Figure 1: A pair of strongly connected bidirected graphs G, H such that H is a subgraph of G and G cannot be obtained from H by adding a single ear.

that the ends of a simple path need not be distinct. As usually, a cyclic edge-simple walk is called a *cycle*. A node-simple cyclic walk is called a *simple* cycle.

We now extend the notions of strong connectivity and ear decomposition to the class of bidirected graphs. We call a bidirected graph G *strongly connected* if its underlying undirected graph is connected and each edge of G is contained in a cycle.

For a bidirected graph G and its subgraph H an *ear* of H w.r.t. G is a path P in G such that: (i) both ends of P are in V_H; (ii) no inner node of P is in V_H; (iii) $E_P \cap E_H = \emptyset$. As earlier, we use notation e to denote the ear consisting of a single edge e.

One can see that unlike the case of directed graphs adding an ear to a strongly connected instance may produce a graph that is not strongly connected (cf. Remark 1.1). Moreover, being restated in terms of bidirected graphs, Theorem 1.2 becomes false. To see this, consider an example depicted in Fig. 1. Both graphs G, H are strongly connected and H can be obtained from G by adding two edges. However, adding only one of these edges does not produce a strongly connected instance.

To overcome this obstacle one needs to allow a pair of ears to be added on certain steps. More formally, consider strongly connected bidirected graphs G and H such that H is a subgraph of G. Also, consider a collection $P_1, \ldots, P_k$ of ears of H w.r.t. G. We denote by $H' := H + P_1 + \ldots + P_k$ the result of adding all ears P_i to H. In particular,

$$V_{H'} := V_H \cup V_{P_1} \cup \ldots \cup V_{P_k}, \qquad E_{H'} := E_H \cup E_{P_1} \cup \ldots \cup E_{P_k}.$$

Consider a strongly connected bidirected graph G and its strongly connected subgraph H. An *ear decomposition* of G starting from H is a sequence of strongly connected subgraphs of G

$$H = G_0, G_1, \ldots, G_{k-1}, G_k = G,$$

where G_{i+1} is obtained from G_i by adding a single ear of G_i w.r.t. G or an edge-disjoint pair of such ears ($0 \leq i < k$). In case G_{i+1} is obtained from G_i by adding only one ear we call it a *single-ear step*; otherwise we are referring to it as a *double-ear step*.

The required generalization of Theorem 1.2 can now be stated as follows:

Theorem 1.3 *For any strongly connected bidirected graph G and an arbitrary strongly connected subgraph H of G there exists an ear decomposition of G starting from H.*

The rest of the paper is organized as follows. In Section 2 we prove a certain special case of Theorem 1.3 (that may be interesting for its own sake). Sections 3 and 4 contain some basic results regarding the so-called skew-symmetric graphs, which are used later in Section 5, where a complete proof of Theorem 1.3 is given. In Section 6 we show how Two Ears Theorem on matching covered graphs can be derived from our results.

2 Two Edges Theorem

Theorem 2.1 *Let G be a strongly connected bidirected graph with all edges standard; let E be a nonempty collection of bidirected edges with both ends in V_G such that each edge in E is nonstandard and $G + E$ is strongly connected. Then there exist a pair of edges $e_1, e_2 \in E$ such that $G + e_1 + e_2$ is also strongly connected.*

Suppose towards contradiction that there exists a graph G and a collection of nonstandard edges E such that $|E| > 2$ and $G + E$ is strongly-connected but no proper subset $E' \subset E$ satisfies this property. In what follows we regard G as a standard directed graph denoting the set of its arcs by A_G. Each edge in E is of two possible kinds: it either enters both ends or leaves them; according to this, we divide E into the subsets E^+ and E^- respectively.

Consider a cycle C in $G + E$ that uses at least one nonstandard edge. Then, C traverses equal number of edges from E^+ and E^-. By assumption of minimality of E, C traverses all edges of E^+ and E^-, and hence $|E^+| = |E^-|$. Put

$$E^+ = \left\{e_1^+, \ldots, e_n^+\right\}, \quad E^- = \left\{e_1^-, \ldots, e_n^-\right\}.$$

We transform G and E in order to make sure that all ends of edges in E are distinct. To this aim we do the following: (i) split each node $v \in V_G$ into a sufficient number of pairs v_i^+, v_i^-; (ii) for each node $v \in V_G$ add arcs (v_i^+, v_j^-) between all possible pairs; (iii) transform each arc $(u, v) \in A_G$ into a collection of arcs (u_i^-, v_j^+) going between all possible pairs. Clearly, this transformation preserves strong connectivity of G.

Finally, each edge $\{u, v\} \in E^+$ (resp. $\{u, v\} \in E^-$) is transformed into an edge $\left\{u_i^+, v_j^+\right\}$ (resp. $\left\{u_i^-, v_j^-\right\}$) of the same type. Here we choose "fresh" values of i, j for each edge to guarantee that all ends are distinct. In what follows we keep the same notation G and E to denote the resulting graph and the resulting set of nonstandard edges.

Recall [8] that for a given nonempty set V a pair (X, Y), $X, Y \subseteq V$, is said to be *crossing* if $X \cap Y \neq \emptyset$, $X \cup Y \neq V$, $X \setminus Y \neq \emptyset$, and $Y \setminus X \neq \emptyset$. A family of sets $\mathcal{F} \subseteq 2^V$ is called *crossing* if $X \cap Y, X \cup Y \in \mathcal{F}$ for every pair of crossing sets $X, Y \in \mathcal{F}$. One can easily see that if $\mathcal{F}$ is crossing, $X, Y \in \mathcal{F}$, $X \cap Y \neq \emptyset$, and $X \cup Y \neq V$, then $X \cap Y, X \cup Y \in \mathcal{F}$. Finally, for a crossing family $\mathcal{F}$, a function $f \colon \mathcal{F} \to \mathbb{R}$ is called *crossing submodular* (on $\mathcal{F}$) if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

holds for all $X, Y \in \mathcal{F}$ such that (X, Y) is a crossing pair.

We need some additional notation. For a set of nodes X denote the set of arcs entering (resp. leaving) X by $\delta^{\text{in}}(X)$ (resp. $\delta^{\text{out}}(X)$). Also $\gamma(X)$ (resp. $\delta(X)$) will denote the set of arcs or edges having both ends (resp. exactly one end) in X.

Put $\varphi(X) := \left|\delta^{\text{in}}(X)\right|$. It is well-known that φ is crossing submodular on 2^V. We consider the following subfamily of 2^V:

$$\mathcal{F}_1 := \{X \subseteq V \mid \varphi(X) = 1\}.$$

Lemma 2.2 *$\mathcal{F}_1$ is a crossing family.*

Proof.

Let (X, Y) be a crossing pair of subsets of V such that $\varphi(X) = \varphi(Y) = 1$. Submodularity of φ implies

$$\varphi(X \cap Y) + \varphi(X \cup Y) \leq \varphi(X) + \varphi(Y) = 2.$$

On the other hand, since $X \cap Y \neq \emptyset$, $X \cup Y \neq V$ and G is strongly connected, one has $\varphi(X \cap Y) \geq 1$ and $\varphi(X \cup Y) \geq 1$. Therefore, $\varphi(X \cap Y) = \varphi(X \cup Y) = 1$ and hence both $X \cap Y$ and $X \cup Y$ are members of $\mathcal{F}_1$. $\square$

Consider a pair of multisets S, T of nodes. By an *S–T collection* we mean a collection of arc-disjoint paths in G such that: (i) each path of $\mathcal{P}$ starts at a node in S and ends at a node in T; (ii) for each $s \in S$ the number of paths from $\mathcal{P}$ starting at s equals the multiplicity of s in S; (iii) for each $t \in T$ the number of paths from $\mathcal{P}$ ending at t equals the multiplicity of t in T.

Let x_i^+, y_i^+ (resp. x_i^-, y_i^-) be the ends of e_i^+ (resp. e_i^-). Consider the sets

$$V^+ := \left\{x_i^+, y_i^+ \mid 1 \leq i \leq n\right\}, \quad V^- := \left\{x_i^-, y_i^- \mid 1 \leq i \leq n\right\}.$$

Let C be a cycle in $G + E$ that traverses each edge of E. Removing edges of E from C we split C into a V^+–V^- collection $\mathcal{P}_0$. Consider an arbitrary index i and the sets $S := \left\{x_1^+, y_1^+\right\}$ and $T := \left\{x_i^-, y_i^-\right\}$. Suppose, there exists an S–T collection. Together with edges e_1^+ and e_i^- these paths form a cycle in $G + E$, contradicting the minimality of E. Therefore, no S–T collection exists. Then, taking strong connectivity of G into account, by a standard max-flow min-cut argument there exists a set $Z_i \in \mathcal{F}_1$ such that $Z_i \cap S_1 = \emptyset$, $T_i \subseteq Z_i$.

We start with sets $Z_1, \ldots, Z_n$ and unite them to construct a collection of inclusion-wise maximum sets $W_1, \ldots, W_m$. More precisely, let H be an undirected graph with nodes $\{1, \ldots, n\}$. For each $1 \leq i < j \leq n$ we add an edge connecting nodes i and j iff $Z_i \cap Z_j \neq \emptyset$. Let $C_1, \ldots, C_m$ be the nodesets of connected components of H. For each i put W_i to be the union of Z_j, $j \in C_i$. Clearly, $W_i \in \mathcal{F}_1$ for all i. From definition of W_i it follows that W_i are pairwise disjoint and $\overline{W} := \bigcup_i W_i$ covers all nodes of V^-.

For each i we say that nodes x_i^+, y_i^+ are *mates*. In particular, x_i^+ is the mate of y_i^+, and y_i^+ is the mate of x_i^+. Same terms are used for x_i^- and y_i^-. A simple inductive argument shows that for each i and $t \in W_i$ the mate of t is also in W_i and $S_1 \cap W_i = \emptyset$.

For a set $X \subseteq V$ put $n^+(X) := |X \cap T^+|$ and $n^-(X) := |X \cap T^-|$. It follows from the existence of $\mathcal{P}_0$ and max-flow min-cut argument that

$$n^+(X) \geq n^-(X) - 1 \qquad \text{for all } X \in \mathcal{F}_1. \tag{1}$$

In view of (1), two cases are possible. First, one may have

$$n^+(W_i) \geq n^-(W_i) \qquad \text{for all } 1 \leq i \leq m. \tag{2}$$

But since W_i are disjoint, (2) implies that $n^+(\overline{W}) \geq n^-(\overline{W})$. However, all nodes in T^- are covered by $\overline{W}$ and at least two nodes in T^+ (namely, x_1^+ and x_1^-) are not covered by $\overline{W}$ — a contradiction.

We may now assume that

$$n^+(W_1) = n^-(W_1) - 1 \tag{3}$$

and W_1 covers the following pairs of mates in T^-:

$$T_1 := \left\{x_1^-, y_1^-, \ldots, x_q^-, y_q^-\right\}. \tag{4}$$

We claim that $q < n$. Suppose $q = n$, then $n^-(W_1) = n$. However, $\left\{x_1^+, y_1^+\right\} \cap W_1 = \emptyset$, thus $n^+(W_1) \leq n - 2$. This contradicts (1).

Let a_0 denote the only arc in G entering W_1 (recall that $W_1 \in \mathcal{F}_1$). $\mathcal{P}_0$ contains a unique path ending in each node of T_1. Put $S_1 := V^+ \cap W_1$; by (3) $|S_1| = |T_1| - 1$ and there exists a unique node $v \in V^+ - S_1$ such that v is connected to some node in T_1, say x_1^-, by the path $P_0 \in \mathcal{P}_0$ that crosses $\delta^{\text{in}}(W_1)$ by a_0. We trace P_0 starting from v until reaching a_0; let R_0 be the suffix of P_0 starting with a_0.

We construct a subcollection of $\mathcal{P}_0$ as follows. Initially, consider the node y_1^-. It is connected by the path $P_1 \in \mathcal{P}_0$ with the node in S_1 that we denote by x_1^+. If $y_1^+ \notin S_1$, then we stop. Otherwise, y_1^+ is connected by the path $Q_1 \in \mathcal{P}_0$ with the node in T_1 that we denote by x_2^-. We now consider its mate y_2^- and proceed it the same way as we did for y_1^-.

In general, on the i-th step we consider the node y_i^- and find the corresponding path $P_i \in \mathcal{P}_0$. Let x_i^+ be the start node of P_i. If $y_i^+ \notin S_1$, we stop. Otherwise, denote by $Q_i \in \mathcal{P}_0$ the path starting at y_i^+. Put x_{i+1}^- to be the end node of Q_i and proceed with the next step.

This procedure eventually halts after, say, l steps yielding a collection of paths

$$P_1, Q_1, \ldots, P_{l-1}, Q_{l-1}, P_l \tag{5}$$

and a node $y_l^+ \in T^+ - S_1$. Note that all these paths are completely contained in $G[W_1]$. Since G is strongly connected, there exists a path Q_l from y_l^+ to x_1^-. This path crosses $\delta^{\text{in}}(W_1)$ and

hence R_0 is a suffix of Q_l. Thus Q_l is arc-disjoint from all paths (5). Put

$$\begin{aligned}
\mathcal{P}' &:= \{P_1, Q_1, \dots, P_{l-1}, Q_{l-1}, P_l, Q_l\}, \\
S' &:= \left\{x_1^+, y_1^+, \dots, x_l^+, y_l^+\right\}, \\
T' &:= \left\{x_1^-, y_1^-, \dots, x_l^-, y_l^-\right\}
\end{aligned}$$

Then $\mathcal{P}'$ is an S'–T' collection that gives rise to a cycle in $G + E$ traversing some but not all edges of E. This, however, contradicts the minimality of E. Proof of Theorem 2.1 is now complete.

3 Skew-Symmetric Graphs

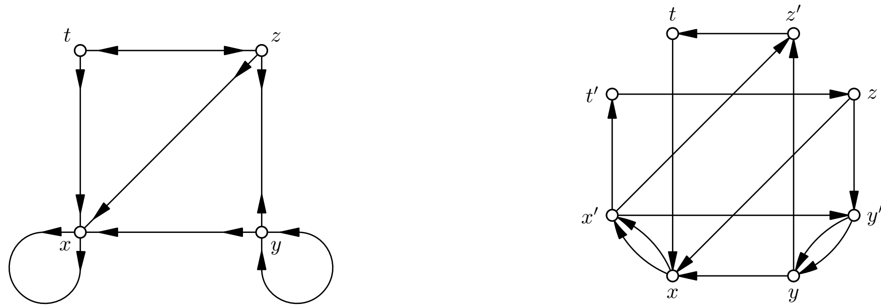
For bidirected graphs there is an alternative (and essentially equivalent) language of *skew-symmetric* graphs. This section contains terminology and some basic facts and explains the correspondence between skew-symmetric and bidirected graphs. For a more detailed survey on skew-symmetric graphs, see, e.g., [9, 4, 5, 2].

A *skew-symmetric graph* is a digraph G endowed with two bijections σ_V, σ_A such that: σ_V is an *involution* on the nodes (i.e., $\sigma_V(v) \neq v$ and $\sigma_V(\sigma_V(v)) = v$ for each node v), σ_A is an involution on the arcs, and for each arc a from u to v, $\sigma_A(a)$ is an arc from $\sigma_V(v)$ to $\sigma_V(u)$. For brevity, we combine the mappings σ_V, σ_A into one mapping σ on $V_G \cup A_G$ and call σ the *symmetry* (rather than skew-symmetry) of G. For a node (arc) x, its symmetric node (arc) $\sigma(x)$ is also called the *mate* of x, and we will often use notation with primes for mates, denoting $\sigma(x)$ by x'.

Observe that if G contains an arc a from a node v to its mate v', then a' is also an arc from v to v' (so the number of arcs of G from v to v' is even and these parallel arcs are partitioned into pairs of mates).

The symmetry σ is extended in a natural way to walks, paths, cycles, and other objects in G. In particular, two walks are symmetric to each other if the elements of the former are symmetric to those of the latter and go in the reverse order: for a walk $P = (v_0, a_1, v_1, \dots, a_k, v_k)$, the symmetric walk $\sigma(P)$ is $(v'_k, a'_k, v'_{k-1}, \dots, a'_1, v'_0)$.

Next we explain the correspondence between skew-symmetric and bidirected graphs (cf. [5, Sec. 2], [2]). For sets X, A, B, we use notation $X = A \sqcup B$ when $X = A \cup B$ and $A \cap B = \emptyset$. Given a skew-symmetric graph G, choose an arbitrary partition $\pi = \{V_1, V_2\}$ of V_G such that $\sigma(V_1) = V_2$. Then G and π determine the bidirected graph $\overline{G}$ with $V_{\overline{G}} := V_1$ whose edges correspond to the pairs of symmetric arcs in G. More precisely, arc mates a, a' of G generate one edge e of $\overline{G}$ connecting nodes $u, v \in V_1$ such that: (i) e goes from u to v if one of a, a' goes from u to v (and the other goes from v' to u' in V_2); (ii) e leaves both u, v if one of a, a' goes from u to v' (and the other from v to u'); (iii) e enters both u, v if one of a, a' goes from u' to v (and the other from v' to u). In particular, e is a loop if a, a' connect a pair of symmetric nodes.



(a) Bidirected graph $\overline{G}$.

(b) Corresponding skew-symmetric graph G.

Figure 2: Related bidirected and skew-symmetric graphs.

Conversely, a bidirected graph $\overline{G}$ determines a skew-symmetric graph G with symmetry σ as follows. Take a copy $\sigma(v)$ of each element v of $\overline{V} := V_{\overline{G}}$, forming the set $\overline{V}' := \left\{\sigma(v) \mid v \in \overline{V}\right\}$. Now put $V_G := \overline{V} \sqcup \overline{V}'$. For each edge e of $\overline{G}$ connecting nodes u and v, assign two "symmetric" arcs a, a' in G so as to satisfy (i)–(iii) above (where $u' = \sigma(u)$ and $v' = \sigma(v)$). An example is depicted in Fig. 2.

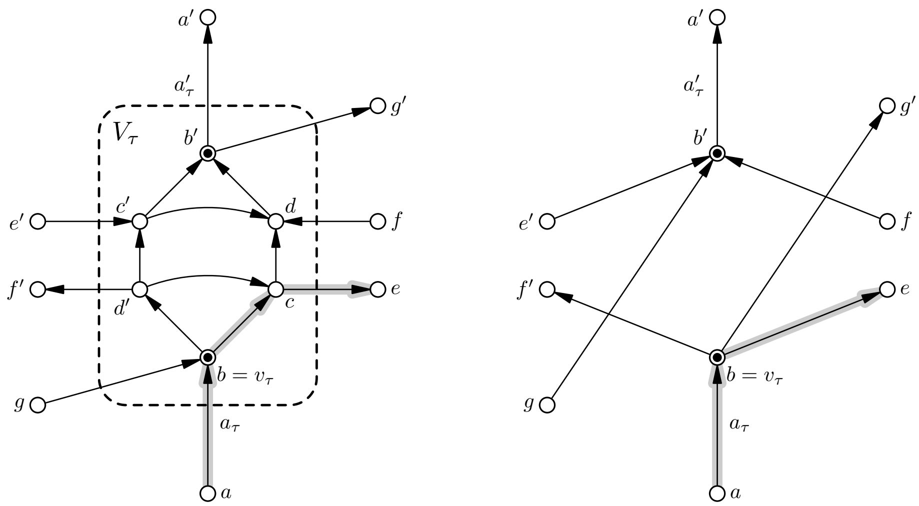
Let X be an arbitrary subset of nodes of a bidirected graph $\overline{G}$. One can modify $\overline{G}$ as follows: for each node $v \in X$ and each edge e incident with v, reverse the direction of e at v. This transformation preserves the set of walks in $\overline{G}$ and thus does not change the graph in essence. We call two bidirected graphs $\overline{G}_1, \overline{G}_2$ *equivalent* if one can obtain $\overline{G}_2$ from $\overline{G}_1$ by applying a number of described transformations.

Remark 3.1 *A bidirected graph generates one skew-symmetric graph, while a skew-symmetric graph generates a number of bidirected ones, depending on the partition π of V_G. The latter bidirected graphs are equivalent.*

Also there is a correspondence between walks in $\overline{G}$ and walks in G. More precisely, let τ be the natural mapping of $V \cup A$ to $\overline{V} \cup \overline{E}$ (obtained by identifying the pairs of symmetric nodes and arcs). Each walk $P = (v_0, a_1, v_1, \ldots, a_k, v_k)$ in G induces the sequence

$$\tau(P) := (\tau(v_0), \tau(a_1), \tau(v_1), \ldots, \tau(a_k), \tau(v_k))$$

of nodes and edges in $\overline{G}$. One can easily check that $\tau(P)$ is a walk in $\overline{G}$ and $\tau(P') = \tau(P)^R$ (where W^R stands for the bidirected walk obtained by passing W in opposite direction). Moreover, for any walk $\overline{P}$ in $\overline{G}$ there is exactly one pre-image $\tau^{-1}(\overline{P})$.



(a) Bud τ in graph G together with a shaded path $\overline{P}$.

(b) Graph G/τ together with a shaded path P.

Figure 3: Buds, trimming, and path restoration. Base and antibase nodes b, b' are marked. Path $\overline{P}$ is a preimage of P.

4 Regular Reachability and Barriers

A path in a skew-symmetric graph is called *regular* if it does not contain a pair of symmetric arcs (while symmetric nodes are allowed). This notion plays an important role since regular paths in a skew-symmetric graph G are exactly the images of paths in the corresponding bidirected graph $\overline{G}$. In this section we state a criterion for the existence of a regular path connecting a pair of symmetric nodes in a skew-symmetric graph.

Consider a skew-symmetric graph G. Let $\tau = (V_\tau, a_\tau)$, $V_\tau \subseteq V_G$, $a_\tau \in A_G$ be a pair such that: (i) $V'_\tau = V_\tau$; (ii) $a_\tau \in \delta^{\rm in}(V_\tau)$; (iii) every node in V_τ is reachable from the head of a_τ by a regular path in $G[V_\tau]$. Then we τ is called a *bud*.

Let v_τ denote the head node of a_τ. The arc a_τ (resp. node v_τ) is called the *base arc* (resp. *base node*) of τ, arc a'_τ (resp. node v'_τ) is called the *antibase arc* (resp. *the antibase node*) of τ. For an arbitrary bud τ we denote its set of nodes by V_τ, base arc by a_τ, and base node by v_τ. An example of a bud is given in Fig. 3(a).

Consider an arbitrary bud τ in a skew-symmetric graph G. By *trimming* τ we mean the following transformation of G: (i) all nodes in $V_\tau - \{v_\tau, v'_\tau\}$ and arcs in $\gamma(V_\tau)$ are removed; (ii) all arcs in $\delta^{\rm in}(V_\tau) - \{a_\tau\}$ are transformed into arcs entering v'_τ (the tails of these arcs are not changed); (iii) all arcs in $\delta^{\rm out}(V_\tau) - \{a'_\tau\}$ are transformed into arcs leaving v_τ (the heads of these arcs are not changed). The resulting skew-symmetric graph is denoted by G/τ. Thus, each arc of the original graph G not belonging to $\gamma(V_\tau)$ has its *image* in the trimmed graph G/τ. Fig. 3

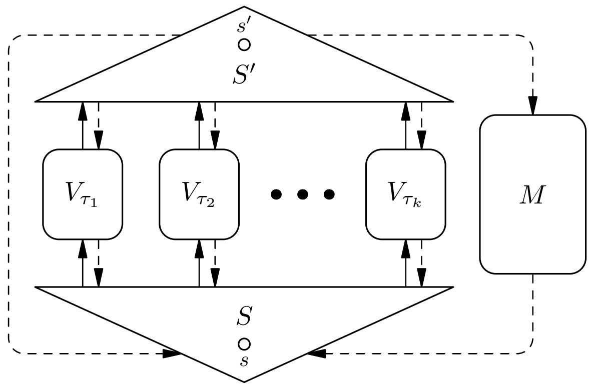


Figure 4: An s-barrier. Solid arcs should occur exactly once, dashed arcs may occur arbitrary number of times (including zero).

gives an example of bud trimming.

Let P be a regular path in G/τ. One can lift this path to G as follows: if P does not contain neither a_τ, nor a'_τ leave P as it is. Otherwise, consider the case when P contains a_τ (the symmetric case is analogous). Split P into two parts: the part P_1 from the beginning of P to v_τ and the part P_2 from v_τ to the end of P. Let a be the first arc of P_2. The arc a leaves v_τ in G/τ and thus corresponds to some arc $\overline{a}$ leaving V_τ in G ($\overline{a} \neq a'_\tau$). Let $u \in V_\tau$ be the tail of a in G and Q be a regular v_τ–u path in $G[V_\tau]$ (existence of Q follows from definition of bud). Consider the path $\overline{P} := P_1 \circ Q \circ P_2$ (here $U \circ V$ denotes the path obtained by concatenating U and V). One can easily show that $\overline{P}$ is regular. We call $\overline{P}$ a *preimage of* P (under trimming G by τ). Clearly, $\overline{P}$ is not unique. An example of such path restoration is shown in Fig. 3: the shaded path $\overline{P}$ on the left picture corresponds to the shaded path P on the right picture.

Let G be a skew-symmetric graph with a designated node s. Suppose we are given a collection of buds $\tau_1, \ldots, \tau_k$ in G together with node sets S and M. Additionally, suppose the following properties hold: (i) collection $\{S, S', M, V_{\tau_1}, \ldots, V_{\tau_k}\}$ forms a partition of V_G with $s \in S$; (ii) no arc goes from S to $S' \cup M$; (iii) no arc connects distinct sets V_{τ_i} and V_{τ_j}; (iv) no arc connects V_{τ_i} and M; (v) the arc a_{τ_i} is the only one going from S to V_{τ_i}. Then we call the tuple $\mathcal{B} = (S, M; \tau_1, \ldots, \tau_k)$ an *s-barrier* ([4], see Fig. 4 for an example).

Theorem 4.1 (Barrier Theorem, [4]) *There exists a regular s–s' path in a skew-symmetric graph G iff there is no s-barrier in G.*

5 Proof of Theorem 1.3

By an inductive argument it is sufficient to prove that given a strongly connected bidirected graph $\overline{G}$ and its strongly connected proper subgraph $\overline{H}$ one can extend $\overline{H}$ to a strongly connected graph by adding one or two edge-disjoint ears of $\overline{H}$ w.r.t. $\overline{G}$. Moreover, one may

assume that no single-ear step is possible at the moment and prove that a double-ear step can be performed in this case.

Consider skew-symmetric graphs G and H that are related to $\overline{G}$ and $\overline{H}$ respectively. Let a_0 be an arc from $A_G - A_H$ that has its tail node u_0 in V_H (such arc exists due to connectivity of underlying undirected graphs of $\overline{G}$ and $\overline{H}$). Since $\overline{G}$ is strongly connected there exists a regular cycle C_0 in G passing through a_0. We follow along this cycle starting from a_0 until reaching the nodeset of H. This way, we construct a path P_0 in G from $u_0 \in V_H$ to, say, $v_0 \in V_H$. The image of P_0 in $\overline{H}$ forms an ear w.r.t. $\overline{G}$.

By assumption that no single-ear step is currently possible, one has no regular path in H from v_0 to u_0. To apply Theorem 4.1 we construct an auxiliary skew-symmetric graph H_0 from H by adding a pair of symmetric nodes s, s' and arcs (s, v_0), (s, u_0'), (v_0', s'), (u_0, s'). It follows that no regular s–s' path exists in H_0 and thus there exists an s-barrier $\mathcal{B}_0 = (\{s\} \cup A, M; \tau_1, \dots, \tau_k)$ in H_0 where $A, M \subseteq V_H$ and τ_i are buds in H_0.

Lemma 5.1 $\mathcal{B} := (A, \emptyset; \tau_1, \dots, \tau_k)$ *is a v_0-barrier in H.*

Proof.

First, suppose that τ_i is not a bud in H. This is only possible if the tail of its base arc a_{τ_i} is s. Hence,

$$\delta_H^{\text{in}}(V_{\tau_i}) = \delta_H^{\text{out}}(V_{\tau_i}) = \emptyset, \tag{6}$$

that a contradiction with connectivity of the underlying undirected graph of $\overline{H}$. Therefore, all τ_i are also buds in H. To see that $v_0 \in A$ note that the only other possibility for v_0 is to be the base node of some bud τ_i. This, however, would again imply (6) and hence is not possible. We also prove that $M = \emptyset$. Indeed, if $\delta^{\text{in}}(M) = \delta^{\text{out}}(M) = \emptyset$, then the underlying undirected graph of $\overline{H}$ is not connected. In case there exists an arc leaving M, from definition of barrier it follows that no regular cycle in H can pass through this arc — again a contradiction. □

Consider the graph $H_1 := H/\tau_1/\dots/\tau_k$ obtained from H by trimming all buds of $\mathcal{B}$. Put $Z := A \cup \{v_{\tau_1}, \dots, v_{\tau_k}\}$ and consider the bidirected graph $\overline{H}_1$ corresponding to H_1 under partition $\{Z, Z'\}$ of V_{H_1} (see Section 1). Since no arc in H_1 connects the sets Z and Z', all edges of $\overline{H}_1$ are standard, so we may regard $\overline{H}_1$ as a digraph isomorphic to $H_1[Z]$. As long as this is not confusing, we make no distinction between $\overline{H}_1$ and $H_1[Z]$.

Lemma 5.2 $H_1[Z]$ *is strongly connected.*

Proof.

The connectivity of the underlying undirected graph follows from this property of $\overline{H}$. Consider an arbitrary arc a of $H_1[Z]$. Consider a regular cycle C passing through a in H; C remains a regular cycle under trimming of all buds in $\mathcal{B}$. The image of C under these trimmings gives rise to a cycle in $H_1[Z]$ that passes through a, as required. □

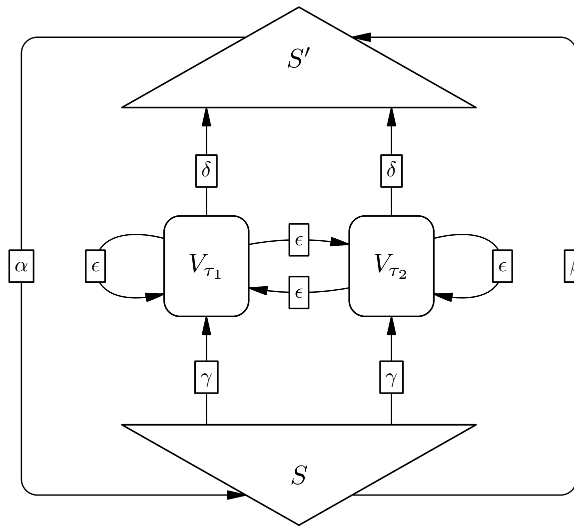


Figure 5: Possible types of ears.

Recall that we originally had the arc $a_0 \in A_G - A_H$ and the regular cycle C_0 passing through a_0. We drop all arcs of C_0 that belong to A_H and thus split C_0 into a collection of ears of H w.r.t. G. Consider an arbitrary such ear P; let u be its start node, and v be its end node. We call P

1. α-*ear* if $u \in S'$, $v \in S$;

2. β-*ear* if $u \in S$, $v \in S'$;

3. γ-*ear* if $u \in S$, $v \in V_{\tau_i}$ for some i;

4. δ-*ear* if $u \in V_\tau$ for some i, $v \in S'$;

5. ϵ-*ear* if $u \in V_{\tau_i}$, $v \in V_{\tau_j}$ for some i, j (possibly $i = j$).

These five cases are depicted in Fig. 5.

Lemma 5.3 *Each ear obtained from C_0 belongs to one these five classes.*

Proof.

Let P be an ear not falling into one of these classes. Due to symmetry, it is sufficient to consider the following two cases: (i) $u, v \in S$; (ii) $u \in V_{\tau_i}$ for some i and $v \in S$. We argue that u is reachable from v by a regular path in H and hence the image of P in $\overline{G}$ is an ear that can be added to $\overline{H}$ without loss of strong connectivity. This contradicts the assumption that no single-ear step is currently possible.

Indeed, in (i) Lemma 5.2 implies that u is reachable from v by a regular path in H_1. By a standard restoration procedure this path can be extended to a regular v–u path in H. In (ii)

v_{τ_i} is reachable from v by a regular path in H_1. Applying restoration procedure and adding a regular v_{τ_i}–u path in $H[V_{\tau_i}]$ one again gets a regular v–u path in H. □

Next, we consider the sequence of ears

$$P_0, P_1, \ldots, P_m \tag{7}$$

obtained from C_0 and construct a collection of nonstandard edges E such that $\overline{H}_1 + E$ is strongly connected. First consider the set of α-ears in (7). Each such ear P (in particular, P_0) goes from a node $u \in S'$ to a node $v \in S$. Construct an edge $\{u, v\}$ (called *backward*) that enters both of its ends and assign the ear P to this edge.

The subsequence of α-ears splits (7) into maximal parts without α-ears. Let $P_i, \ldots, P_j$ be any of these parts. The part gives rise to a nonstandard edge as follows. In case P_i is β- or γ-ear put x to be the start node of P_i. Otherwise (P_i is δ- or ϵ-ear) put x to be the base node of the bud containing the start node of P_i. Similarly, consider P_j. In case P_j is β- or δ-ear put y to be the end node of P_i. Otherwise (P_j is γ- or ϵ-ear) put y to be the antibase node of the bud containing the end node of P_j. Construct an edge $\{x, y\}$ (called *forward*) that leaves both of its ends and assign the sequence of ears $P_i, \ldots, P_j$ to this edge.

As a result, we get a collection of nonstandard edges E. All these edges belong to a cycle in $\overline{H}_1 + E$ obtained from C_0 as follows:

(8) All arcs in $\gamma(V_{\tau_i})$, $i = 1, \ldots, k$ are dropped. Each α-ear Q in (7) is replaced by the arc corresponding to the backward edge assigned to Q. Each maximal sequence $P_i, \ldots, P_j$ of β-, γ-, δ-, and ϵ-ears is replaced by the arc corresponding to the forward edge assigned to $P_i, \ldots, P_j$. Finally, the bidirected image in $\overline{H}_1 + E$ is taken by merging mates of nodes and arcs.

Hence, $\overline{H}_1 + E$ is strongly connected. Now Theorem 2.1 implies the existence of a pair of edges $e_1, e_2 \in E$ (where e_1 is forward and e_2 is backward) such that $\overline{H}_1 + e_1 + e_2$ is strongly connected. Our final task is to replace these edges by a pair of ears of $\overline{H}$ w.r.t. $\overline{G}$.

A trivial part is to deal with e_2 since it corresponds to a single ear in (7). In contrast, e_1 may correspond to a number of ears. We first prove the following auxiliary statement:

Lemma 5.4 *Consider an arbitrary strongly connected digraph and five nodes a, b, x, y, z in it. Suppose there exists an $\{a, b\}$–$\{x, y\}$ collection of paths. Then, there exists an $\{a, b\}$–$\{x, z\}$ or $\{a, b\}$–$\{z, y\}$ collection.*

Proof.

We may assume that there exist an a–x path P and a b–y path Q that are arc-disjoint. Consider an arbitrary a–z path R. We follow it backwards starting from z and stop either when reaching a or encountering an arc from A_P or A_Q. If a is reached, then $\{Q, R\}$ is a desired $\{a, b\}$–$\{z, y\}$ collection. If an arc from P is encountered, then we get an $\{a, b\}$–$\{z, y\}$

collection by taking path Q and parts of paths P, R. Finally, if an arc from P is encountered, an $\{a,b\}$–$\{x,z\}$ collection is obtained by taking path P and parts of paths Q, R. $\square$

To complete the proof we now proceed iteratively as follows. We maintain a pair of non-standard edges e_1, e_2 (e_1 is forward, e_2 is backward). Edge e_2 is assigned a α-ear from (7); let us denote this ear by Q. Edge e_1 is assigned a sequence $P_i, \ldots, P_j$ of ears from (7). The following invariant holds: there exists a regular cycle C in the skew-symmetric graph $H + (Q + Q') + (P_i + P_i') + \ldots + (P_j + P_j')$ that passes through all arcs of $Q, P_i, \ldots, P_j$ in this order. Moreover, C gives rise to a cycle $\overline{C}_1$ in $\overline{H}_1 + e_1 + e_2$ according to (8).

Let $\{a,b\}$ be the multiset of ends of e_2 and $\{x,y\}$ be the multiset of ends of e_1. Due to symmetry, we may assume that x is the start node of P_i. (Hereinafter we identify nodes of $\overline{H}_1$ with those of $H_1[Z]$ and $H[Z]$.) By dropping edges e_1, e_2 from $\overline{C}_1$ one gets an $\{a,b\}$–$\{x,y\}$ collection of paths in H_1. In case $i = j$, a unique ear corresponds to e_1 and hence we are done. Otherwise we change e_1, i, j so as to reduce the number of ears assigned to e_1. Consider P_i; it cannot be a β- or δ-ear since that would imply $i = j$. Hence, two cases are possible.

If P_i is a γ-ear then put z to be base node of the bud containing the end node of P_i. Apply Lemma 5.4 and replace $\{x,y\}$ by either $\{z,y\}$ or $\{x,z\}$. In the former case put $i := i+1$, in the latter put $j := i$. Also, update the cycle C and the edge e_1 to reflect the changes in its ends and proceed with the next iteration.

Now suppose P_i is ϵ-ear with the start node in the nodeset of a certain bud, say τ. In this case $x = v_\tau$. The cycle C enters V_τ by the arc a_τ, uses some arcs from $\gamma_H(V_\tau)$, and then leaves V_τ by P_i. We make sure that P_i is the only ear assigned to e_1 that leaves V_τ. If it is not true then we replace i by the largest index k in the range $i, \ldots, j$ such that P_k leaves V_τ. The cycle C and the edge e_1 are updated accordingly.

Then, let η be the bud whose nodeset contains the end node of P_i; put $z := v_\eta$. Like earlier, we apply Lemma 5.4 and and replace $\{x,y\}$ by either $\{z,y\}$ or $\{x,z\}$. In the former case put $i := i+1$, in the latter put $j := i$. As before, update the cycle C and the edge e_1 to reflect the changes in its ends x, y and proceed with the next iteration.

Once iterations are complete, we get a single ear P_i is assigned to e_1. The bidirected images of P_i, Q in $\overline{G}$ form the desired pair of ears of $\overline{H}$ w.r.t. $\overline{G}$. The proof of Theorem 1.3 is now complete.

6 Application to Matching Covered Graphs

Recall [7] that a *perfect matching* M in an undirected graph G is a set of edges such that each node $v \in V_G$ is incident with exactly one edge in M. An undirected graph is called *matching covered* if every edge $e \in E_G$ is contained in a perfect matching. An path in G is called *alternating* w.r.t. M if it consists of an alternating sequence of edges from M and $E_G - M$.

A subgraph H of G is called *elastic* (w.r.t. G) if $G[V_G - V_H]$ has a perfect matching. By an *ear* of H w.r.t. G we mean a simple path P of odd length in G such that: (i) ends of P are

distinct and are contained in V_H; (ii) no inner node of P is contained in V_H; (iii) $E_P \cap E_H = \emptyset$. The result of adding P to H is denoted by $H + P$ and is defined in a natural way.

An *ear decomposition* of a matching covered graph G starting from its elastic matching covered subgraph H is a sequence of elastic (w.r.t. G) matching covered subgraphs of G

$$H = G_0, G_1, \ldots, G_{k-1}, G_k = G,$$

where G_{i+1} is obtained from G_i by adding a single ear of G_i w.r.t. G or a node-disjoint pair of such ears ($0 \le i < k$).

We use Theorem 1.3 to derive the following result of Lovász and Plummer:

Theorem 6.1 *For any matching covered graph G and an arbitrary elastic subgraph H of G there exists an ear decomposition of G starting from H.*

Proof.

It is sufficient to prove that for a matching covered graph G and its elastic matching covered proper subgraph H the latter one can be extended to an elastic matching covered graph by adding one or two ears of H w.r.t. G.

Consider a perfect matching M_G in G such that $M_H := M_H \cap E_H$ is a perfect matching in H (existence of M_G follows from elasticity of H). Then G, M_G generate the bidirected graph $\overline{G}$ as follows. Each edge $e \in E_G - M_G$ is directed so as to leave both of its ends. Each edge $e = \{u, v\} \in M$ is transformed into a pair of parallel edges e_1, e_2 connecting nodes u, v. The former one enters u and v; the latter one leaves u and v. Edges e_2 are called *auxiliary.* A similar construction applied to H, M_H yields the bidirected subgraph $\overline{H}$ of $\overline{G}$.

We prove that $\overline{G}$ is strongly connected (the same argument also applies to $\overline{H}$). For each $e \in E_G$ the edges e_1, e_2 form a cycle in $\overline{G}$. So it remains to consider edges $e \in E_G - M_G$. From definition of matching covered graph and simple facts regarding perfect matchings it follows that there exists an alternating cycle in G w.r.t. M_G that passes through e. This cycle in G gives rise to a desired cycle in $\overline{G}$ passing through e.

Consider an arbitrary ear $\overline{P}$ of $\overline{H}$ w.r.t. $\overline{G}$ and its image P in G (obtained by dropping directions of edges and merging e_1, e_2 into e, where $e \in M_G$). Suppose $\overline{P}$ contains an auxiliary edge e_2 (corresponding to the edge $e = \{u, v\} \in M_G$). It follows that both ends of $\overline{P}$ are contained in the set $\{u, v\}$. Hence, $\{u, v\} \subseteq V_H$ and $e \in M_H$. Thus, $e_2 \in E_{\overline{H}}$, which is a contradiction. It is now easy to see that P is an alternating path in G w.r.t. M_G with first and last edges in $E_G - M_G$. Therefore, it has an odd length.

We apply Theorem 1.3 to $\overline{H}, \overline{G}$ to get a collection $\overline{\mathcal{P}}$ of at most two edge-disjoint ears of $\overline{H}$ w.r.t. $\overline{G}$ such that adding all ears of $\overline{\mathcal{P}}$ to $\overline{H}$ one gets a strongly connected graph $\overline{H}' := \overline{H} + \overline{\mathcal{P}}$. We may assume that $\left|\overline{\mathcal{P}}\right|$ is minimal and hence there exists a cycle $\overline{C}$ in $\overline{H}'$ that passes through all ears from $\overline{\mathcal{P}}$. The image C of $\overline{C}$ in G is an alternating cycle w.r.t. M_G. Each alternating cycle is simple and thus all nodes of ears in $\overline{\mathcal{P}}$ are distinct, as required.

It remains to show that the graph H' (obtained from H by adding the images of ears from $\overline{\mathcal{P}}$) is elastic and matching covered. The former property follows from the fact that

$M_G \cap \gamma(V_G - V_{H'})$ is a perfect matching in $G[V_G - V_{H'}]$. The latter property is due to the strong connectivity of $\overline{H}'$. $\square$

7 Acknowledgments

The author is thankful to Alexander Karzanov for constant attention, collaboration, and many insightful discussions.

References

[1] Maxim A. Babenko. Acyclic bidirected and skew-symmetric graphs: Algorithms and structure. In *CSR 2006, LNCS 3967*, pages 23–34, 2006.

[2] Maxim A. Babenko and Alexander V. Karzanov. Free multiflows in bidirected and skew-symmetric graphs. *To appear in a special issue of Discrete Applied Mathematics*, 2005.

[3] J. Edmonds and E. L. Johnson. Matching, a well-solved class of integer linear programs. *Combinatorial Structures and Their Applications*, pages 89–92, 1970.

[4] Andrew V. Goldberg and Alexander V. Karzanov. Path problems in skew-symmetric graphs. *Combinatorica*, 16(3):353–382, 1996.

[5] Andrew V. Goldberg and Alexander V. Karzanov. Maximum skew-symmetric flows and matchings. *Mathematical Programming*, 100(3):537–568, 2004.

[6] E. L. Lawler. *Combinatorial Optimization: Networks and Matroids.* Holt, Reinhart, and Winston, NY, 1976.

[7] L. Lovász and M. D. Plummer. *Matching Theory.* North-Holland, NY, 1986.

[8] A. Schrijver. *Combinatorial Optimization*, volume A. Springer, Berlin, 2003.

[9] W. T. Tutte. Antisymmetrical digraphs. *Canadian J. Math.*, 19:1101–1117, 1967.